toepassing van voorgaande op

$\boxed{\text{POSITIEVE MACHTENREEKSEN}}$

C

stel $\sum_{m=1}^{+\infty} a_m (z - z_0)^m$ abs. conv. voor $\underbrace{|z - z_0| < R}_{z \in \mathring{B}(z_0; R)}$

neem

$K \subset \mathring{B}(z_0; R)$
compact

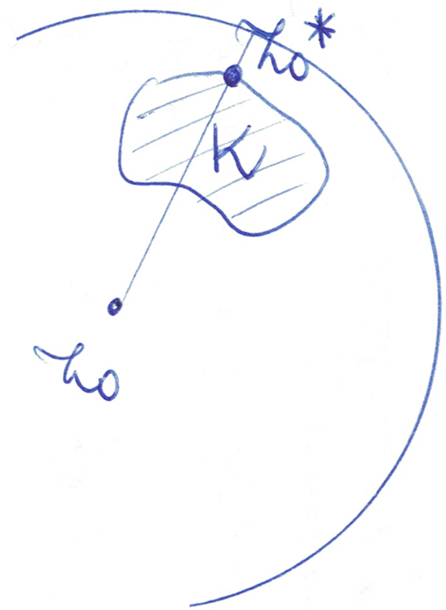
en stel

$|z_0^* - z_0| = \max_{z \in K} |z - z_0|$

↓

punt verst van z_0

dan is : $|z_0^* - z_0| < R$



<u>Dus</u> $\sum_{n=0}^{+\infty} |a_n| |z - z_0|^n$

$$\leq \underbrace{\sum_{n=0}^{+\infty} |a_n| |z_0^* - z_0|^n}_{\text{convergente majorante}} \quad (\to \text{M-test})$$

want

(d'Alembert) $\underbrace{\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|}_{= R^{-1}} \underbrace{|z_0^* - z_0|}_{< R} < 1$

$\Rightarrow$ uniform convergent op K, $\forall K \subset \mathring{B}(z_0; R)$ compact

$\Rightarrow$ $\boxed{f \text{ continu}}$ <u>volgende vraag</u>: f afl?

afleiden v/e PMR

$$f(z) = \sum_{m=0}^{+\infty} a_m (z-z_0)^m \quad \forall z \in \mathring{B}(z_0; R)$$

$\downarrow$ afl.

$$\underbrace{g(z)}_{f'(z)} = \underbrace{\sum_{m=1}^{+\infty} m\, a_m (z-z_0)^{m-1}}_{\substack{\text{PMR} \\ \text{uniform conv} \\ \text{op de compacta} \\ \text{van } \mathring{B}(z_0; R')}}$$

? welk gebied?

ZELFDE!

$$\text{met } R' = \lim_{m \to +\infty} \left| \frac{m}{m+1} \cdot \frac{a_m}{a_{m+1}} \right| = R$$

Dus $f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$

$\downarrow$

onbeperkt afleidbaar in $\mathring{B}(z_0; R)$

met:

- $f'(z) = \sum_{n=1}^{+\infty} n\, a_n (z - z_0)^{n-1}$
- $f''(z) = \sum_{n=2}^{+\infty} n(n-1)\, a_n (z - z_0)^{n-2}$
- ...

$$\Rightarrow f^{(k)}(z) = \sum_{n=k}^{+\infty} n(n-1)\cdots(n-k+1)\, a_n (z - z_0)^{n-k}$$

let op: bij elke stap blijft $\mathring{B}(z_0; R)$ maar wel rand afz. onderzoeken

merk op: Stel $x = x_0$ in $f^{(k)}(x)$
dan

$$f^{(k)}(x_0) = \underbrace{\big(k(k-1)\cdots 1\big)}_{=\,k!}\, a_k$$

$$\Rightarrow \boxed{a_k = \frac{f^{(k)}(x_0)}{k!}}$$

CONCLUSIE

als $f(x) = \sum_{m=0}^{+\infty} a_m (x-x_0)^m$ in $\mathring{B}(x_0; R)$

naam: holomorfe / analytische functie in $\mathring{B}(x_0; R)$

dan is $f(x)$ onbeperkt afleidbaar in $\mathring{B}(x_0; R)$

en $a_m = \dfrac{f^{(m)}(x_0)}{m!}$ dus

$$\boxed{f(x) = \sum_{m=0}^{+\infty} \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m}$$

TAYLORREEKS van f

<u>VB 1</u> $\sum_{n=0}^{+\infty} z^n = \frac{1}{1-z}$ voor $z \in \mathring{B}(0;1)$

Stel $z = -x^2$ dan $\boxed{\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}}$
$x \in]-1,1[$

primitiveer beide leden:

$$\arctan(x) + C = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

C ↓ primitiverings constante

bepaal via $\boxed{x=0}$: $\arctan(0) + C = 0$

$\Rightarrow \boxed{C=0}$

<u>Dus</u>

$$\arctan(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \forall x \in]-1,1[$$

? $\boxed{x = \pm 1}$ (randpunten)

substitueer in de reeks

$$\pm \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} \longrightarrow$$ convergente wisselreeks wegens Leibniz

niet absoluut conv. (OEF)

$\downarrow$

betrekkelijk convergent voor $x = \pm 1$

VB2 $$f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

* convergentie $$\lim_{n \to +\infty} \left| \frac{\cancel{x^{n+1}}}{(n+1)!} \cdot \frac{n!}{\cancel{x^n}} \right| = 0 < 1 \quad \forall x$$

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Dus $f(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$ $\forall z \in \mathbb{C}$
overal convergent

$$\Rightarrow f'(z) = \sum_{n=1}^{+\infty} \frac{n z^{n-1}}{n!}$$

$$\Rightarrow f'(z) = \sum_{n=1}^{+\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n'=0}^{+\infty} \frac{z^{n'}}{n'!}$$

[stel $n' = n - 1$
vervang nadien terug door n

Maar
$$f'(z) = f(z) \Rightarrow \boxed{f(z) = \underset{\substack{\| \\ 1 \\ \text{via } z=0}}{C} \exp(z)}$$

pas toe met $\boxed{z = ix}$, $x \in \mathbb{R}$
dan

$$\exp(ix) = \sum_{m=0}^{+\infty} \frac{(ix)^m}{m!} = \cos(x) + i\sin(x)$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}_{R_1(x)} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}_{R_2(x)}$$

$$R_1(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos(x)$$

$$R_2(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin(x)$$

$\forall x \in \mathbb{R}$

met $R_1'(x) = -R_2(x)$ en $R_2'(x) = R_1(x)$