UNIVERSITEIT GENT

FACULTY OF ECONOMICS

AND BUSINESS ADMINISTRATION

OPERATIONS RESEARCH

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Course outline

CHAPTER

LINEAR PROGRAMMING – THE SIMPLEX METHOD

Course Outline

Deterministic modelling	Linear programming	Modelling Solving: The Simplex Method Duality Theory Sensitivity Analysis Special Cases: Transportation, Assignment and Transshipment Problems
	Network optimization	Modelling and solving
	Integer programming	Modelling Solving Constraint Programming
	Dynamic programming	Modelling and Solving
	Nonlinear programming	Modelling and Solving
Probabilistic modelling	Decision making under uncertainty	Decision Analysis Game Theory Markov Chains
	Queueing Theory	Modelling and Solving

LINEAR PROGRAMMING PROBLEMS

THE SIMPLEX METHOD

Outline

- Introduction
- Linear Programming Formulation
- Solution Method
  - The Graphical Solution Method
  - The Spreadsheet Solution Method
  - The Simplex Method
- Duality Theory
- Sensitivity Analysis

Outline

- The Simplex Method: The Principles
- Setting Up the Simplex Method
  - Standard Form
  - Canonical Form
  - The Simplex Tableau
- The Algebra of the Simplex Method
- The Simplex Method
  - Determine the entering variable
  - Determine the leaving variable
  - Generate the next simplex tableau
- Special cases
- Two-Phase Method

The Simplex Method: The Principles

- Example

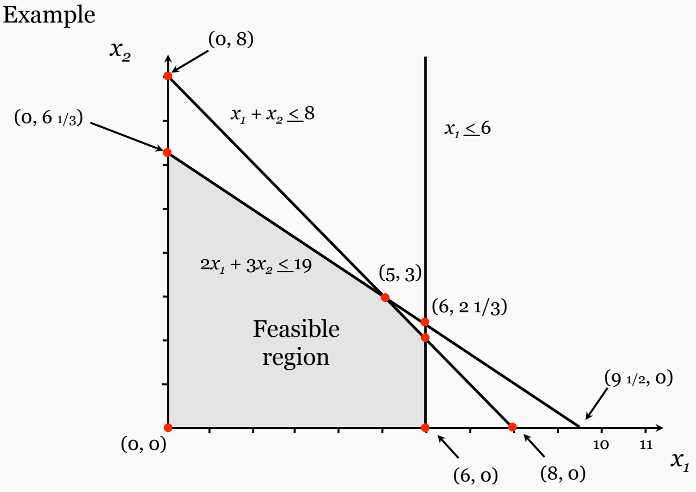
Maximize $Z = 5x_1 + 7x_2$

subject to

$$x_1 \leq 6$$
$$2x_1 + 3x_2 \leq 19$$
$$x_1 + x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

The Simplex Method: The Principles

- Example



The Simplex Method: The Principles

- Terminology
  - Constraint boundary

    $x_1 = 6$; $2x_1 + 3x_2 = 19$; $x_1 + x_2 = 8$; $x_1 = 0$; $x_2 = 0$

  - Corner-point solutions
    - For a linear programming problem with n decision variables, each of the corner-point solutions lies at the intersection of n constraint boundaries
    - Corner-point feasible solutions (CPF solutions)
      - The points that lie on the corners of the feasible region (cfr (0, 0); (6, 0); (6, 2); (5, 3); (0, 6 1/3))
    - Corner-point infeasible solutions
      - Cfr (0, 8); (8, 0); (6, 2 1/3)

The Simplex Method: The Principles

- Terminology
  - Adjacent corner-point solutions
    - For a linear programming problem with n decision variables, two CPF solutions are adjacent to each other if they share $n - 1$ constraint boundaries.
    - They are connected by a line segment or an edge of the feasible region (on the same shared constraint boundaries).
    - E.g. (0,0) and (6, 0) share the constraint boundary $x_2 = 0$.
    - Very useful concept in checking the optimality of a CPF solution.

The Simplex Method: The Principles

- Optimality Test
  - If a CPF solution has no adjacent CPF solutions that are better, then it must be an optimal solution.

E.g. (5, 3) must be an optimal solution

(5, 3) > Z = 46

(6, 2) > Z = 44

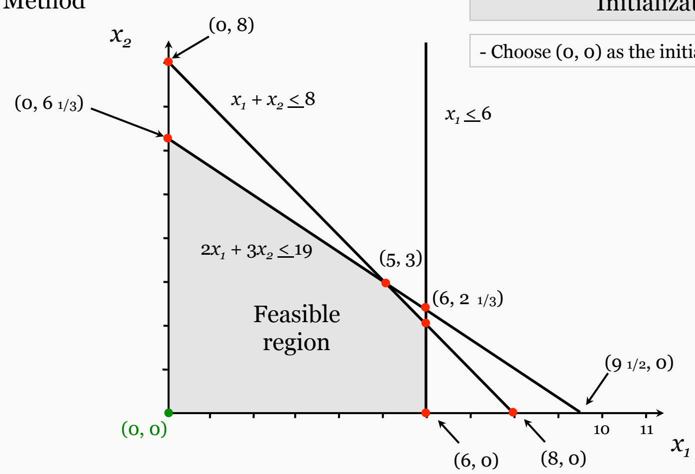
(0, 6 1/3) > Z = 44 1/3

The Simplex Method: The Principles

- Method

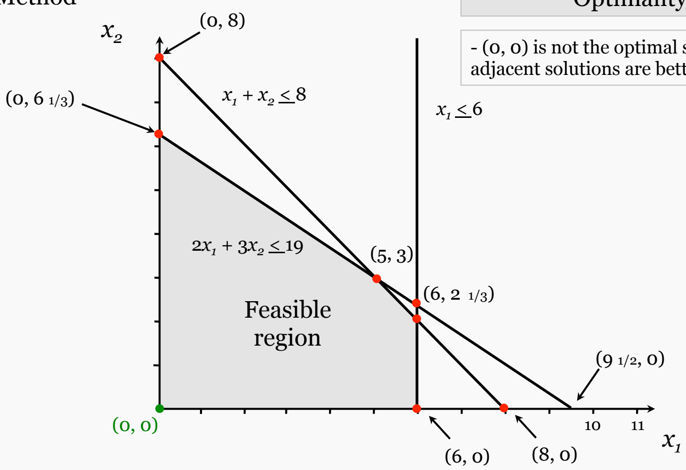
Initialization

- Choose (0, 0) as the initial CPF solution



The Simplex Method: The Principles

- Method

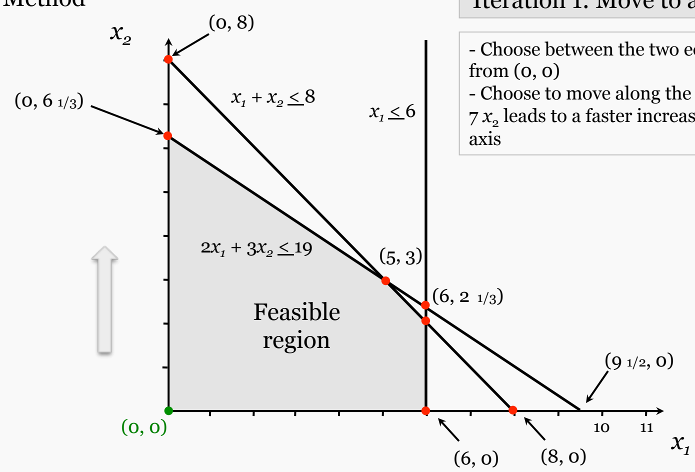


Optimality Test

- (0, 0) is not the optimal solution as adjacent solutions are better.

The Simplex Method: The Principles

- Method



Iteration 1: Move to an adjacent CPF

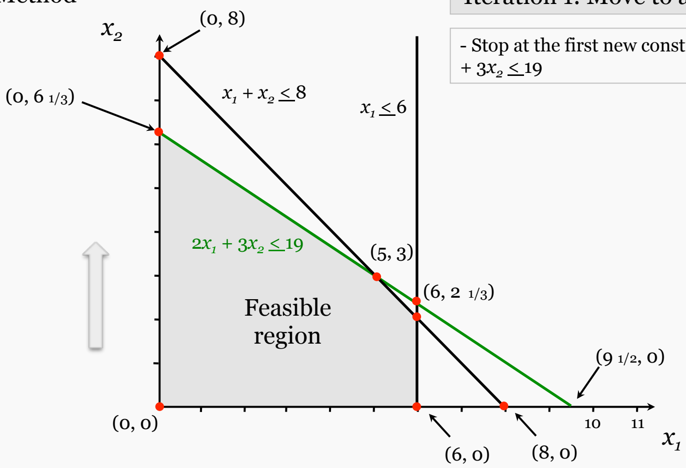
- Choose between the two edges that emanate from (0, 0)
- Choose to move along the x_2 axis as Z = 5 x_1 + 7 x_2 leads to a faster increase moving up the x_2 axis

The Simplex Method: The Principles

- Method

Iteration 1: Move to an adjacent CPF

- Stop at the first new constraint boundary $2x_1 + 3x_2 \leq 19$

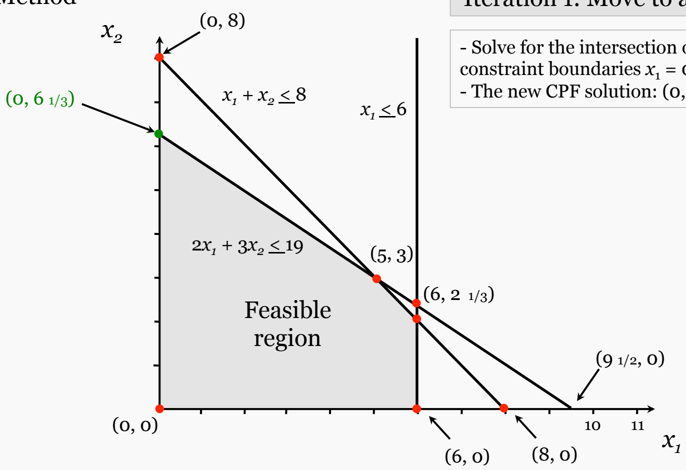


The Simplex Method: The Principles

- Method

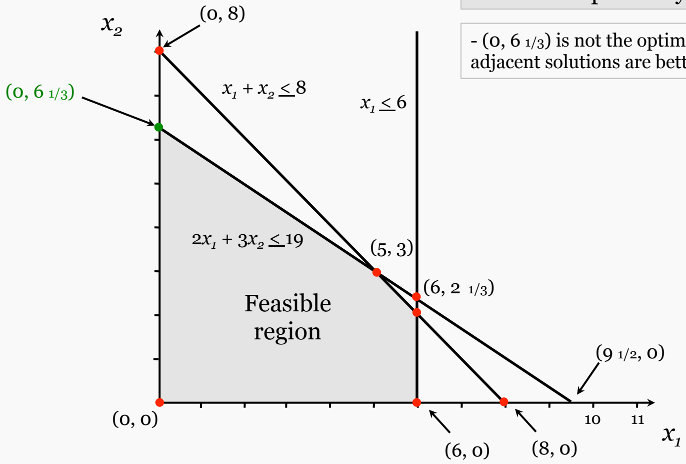
Iteration 1: Move to an adjacent CPF

- Solve for the intersection of the new set of constraint boundaries $x_1 = 0$ and $2x_1 + 3x_2 \leq 19$
- The new CPF solution: (0, 6 1/3)



The Simplex Method: The Principles

- Method

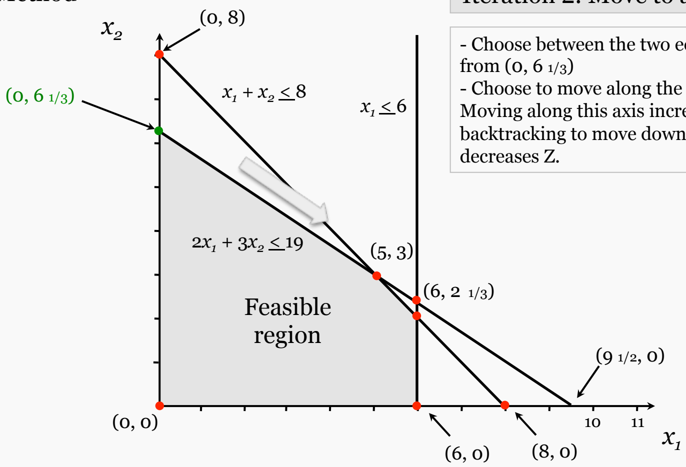


Optimality Test

- (0, 6 1/3) is not the optimal solution as adjacent solutions are better.

The Simplex Method: The Principles

- Method

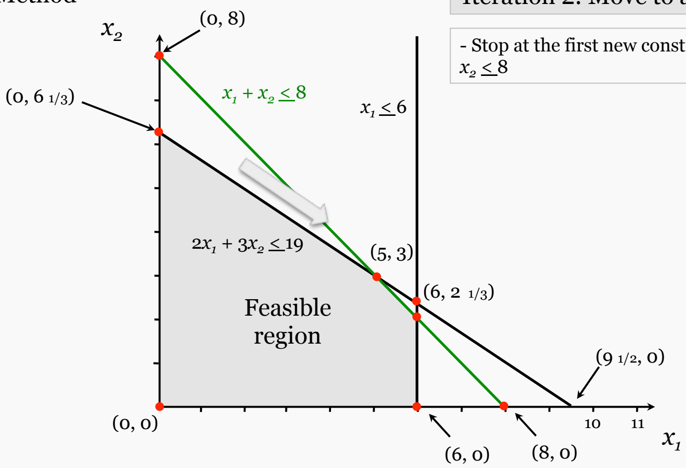


Iteration 2: Move to an adjacent CPF

- Choose between the two edges that emanate from (0, 6 1/3)
- Choose to move along the $2x_1 + 3x_2 \leq 19$. Moving along this axis increases Z whereas backtracking to move down the x_2 axis decreases Z.

The Simplex Method: The Principles

- Method

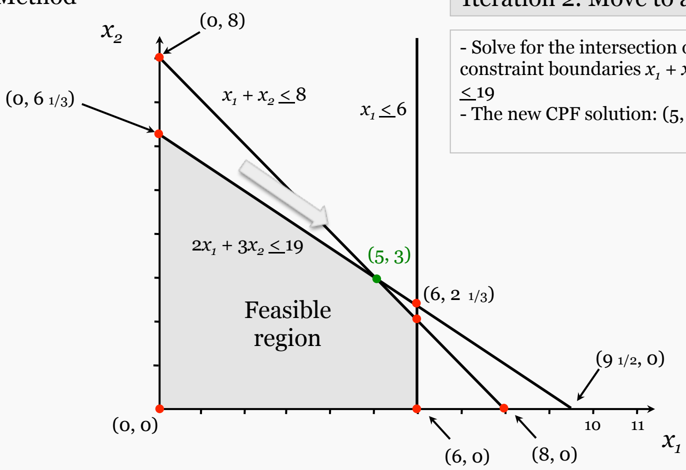


Iteration 2: Move to an adjacent CPF

- Stop at the first new constraint boundary $x_1 + x_2 \leq 8$

The Simplex Method: The Principles

- Method

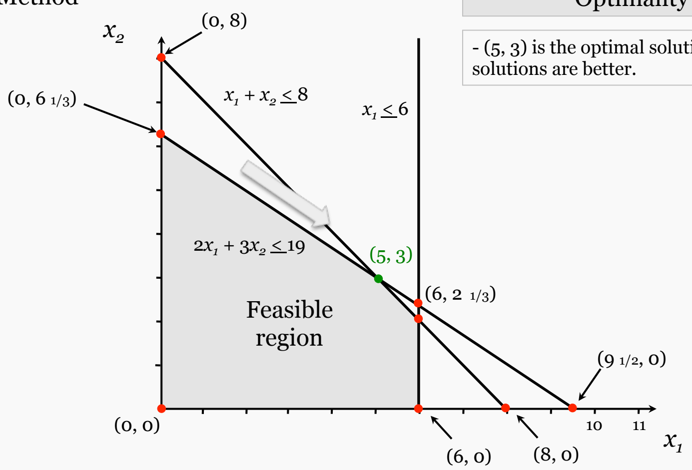


Iteration 2: Move to an adjacent CPF

- Solve for the intersection of the new set of constraint boundaries $x_1 + x_2 \leq 8$ and $2x_1 + 3x_2 \leq 19$
- The new CPF solution: (5, 3)

The Simplex Method: The Principles

- Method

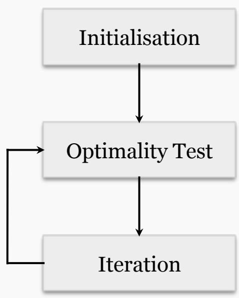


Optimality Test

- (5, 3) is the optimal solution as no adjacent solutions are better.

The Simplex Method: The Principles

- Key Solution Concepts
  - Solution concept 1: The simplex method focuses solely on CPF solutions.
  - Solution concept 2: The simplex method is an iterative algorithm with the following structure.



Initialisation: Set up to start iterations, including finding an initial CPF solution

Optimality Test: Is the current CPF solution optimal? If yes, STOP

Iteration: Perform an iteration to find a better CPF solution

The Simplex Method: The Principles

- Key Solution Concepts
  - Solution concept 3: Whenever possible, the initialisation of the simplex method chooses the origin (all the decision variables are equal to zero) to be the initial CPF solution.
  - Solution concept 4: Given a CPF solution, it is much easier to gather information about the adjacent CPF solutions than about other CPF solutions.

The Simplex Method: The Principles

- Key Solution Concepts
  - Solution concept 5: The choice of the next adjacent CPF solution is dependent on the rate of improvement in Z that would be obtained by moving along the edge.
    - Among the edges with a positive rate of improvement in Z, the edge with the largest rate of improvement is chosen.
  - Solution concept 6: Optimality test boils down to checking the rate of improvement along the edges.
    - A positive rate of improvement in Z implies that the adjacent CPF solution is better than the current CPF solution.
    - A negative rate of improvement in Z implies that the adjacent CPF solution is worse.
    - If no rates of improvement in Z are positive, the current CPF solution is optimal.

Setting Up the Simplex Method

- Preparation and initialization
  - Step 1. Put the original problem formulation into standard form
  - Step 2. Select the origin as initial basic solution
  - Step 3. Put the standard form into canonical form

Setting Up the Simplex Method

- Standard Form
  - Convert the functional inequality constraints to equivalent equality constraints by transposing the LP formulation to standard form.
  - A linear program in which all the variables are non-negative and all the constraints are equalities is said to be in standard form (or augmented form).
  - Standard form is attained by
    - adding slack variables to 'less than or equal to' constraints
    - subtracting surplus variables from 'greater than or equal to' constraints.
  - Slack and surplus variables represent the difference between the left and right sides of the constraints.
    - If a slack variable is equal to 0, the current solution lies at the constraint boundary of the functional constraint.
    - If a slack variable is greater than 0, the current solution lies on the feasible side of the constraint boundary
  - Slack and surplus variables have objective function coefficient equal to 0.

Setting Up the Simplex Method

- Mathematical Problem Formulation

Maximize $c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2$$
$$\ldots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m$$
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

Setting Up the Simplex Method

- Standard Mathematical Problem Formulation

Maximize $c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + 0\, s_1 + 0\, s_2 + \ldots + 0\, s_m$

subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + s_1 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + s_2 = b_2$$
$$\ldots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n + s_m = b_m$$
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, s_1 \geq 0, \ldots, s_m \geq 0$$

Slack variables

Setting Up the Simplex Method

- Standard Form
  - Example LP Formulation

Maximize $3x_1 + 5x_2$

subject to

$$
\begin{aligned}
x_1 \quad\quad & \le 4 \\
2x_2 & \le 12 \\
3x_1 + 2x_2 & \le 18 \\
x_1 \ge 0, \quad x_2 & \ge 0
\end{aligned}
$$

Setting Up the Simplex Method

- Standard Form
  - Example LP Formulation

Maximize $100x_1 + 200x_2$

subject to

$$
\begin{aligned}
2x_1 + 3x_2 & \le 2\,000 \\
x_1 \quad\quad & \ge 60 \\
x_2 & \le 720 \\
x_2 & \ge 0
\end{aligned}
$$

Setting Up the Simplex Method

- Standard Form
  - Example LP Formulation

Minimize	$100\,x_A$ +	$80\,x_B$	
subject to	$2\,x_A$ -	x_B	≥ 0
	x_A +	x_B	$\geq 1\,000$
	x_A ,	x_B	≥ 0

Setting Up the Simplex Method

- Basic Solutions
  - A basic solution is an augmented corner-point solution (and includes slack variable values).
  - A basic feasible solution is an augmented CPF solution.
    - E.g. corner point (6, 0) refers to the basic solution (6, 0, 0, 7, 2)
  - The LP problem has:
    - $(n + m)$ variables: n decision variables + m slack variables
    - m equations

    } Degrees of freedom = n
  - In a basic solution, there is one basic variable for each functional constraint. The number of nonbasic variables equals the total number of variables minus the number of functional constraints. All other variables, the non-basic variables, are zero.

Setting Up the Simplex Method

- Basic Solutions
  - If we set n of the (n + m) variables to zero, then we have a system with m (basic) variables and m equations, which can be solved using linear algebra. The resulting solution is called a basic solution.
  - Number of possible basic solutions: $\binom{n+m}{m}$
  - The *n* variables equal to zero are the 'non-basic' variables, the *m* non-zero variables are the 'basic' variables.

Setting Up the Simplex Method

- Basic Solutions
  - Putting non-basic variables to zero takes us to a corner of the feasible region (i.e. where the optimal solution might be found).
  - Giving non-zero values to non-basic variables takes us away from the corners of the feasible region, which is not useful.
  - Example

Setting Up the Simplex Method

- Adjacent Basic Solutions
  - Two basic solutions are adjacent if all but one of their nonbasic variables are the same. E.g. (6, 0, 0, 7, 2) and (0, 0, 6, 19, 8).
  - Moving from one basic solution to another involves switching one variable from nonbasic to basic and vice versa for another variable.

    E.g. From (0, 0, 6, 19, 8) to (6, 0, 0, 7, 2) involves switching x_1 from nonbasic to basic.
  - Example: (6, 0, 0, 7, 2) is a basic solution
    - Augment the CPF solution (6, 0)
    - Choose x_2 and s_1 as nonbasic variables and set these variables equal to zero. Solve the following set of equations to find the corresponding basic solution:

(1)	$x_1 + s_1 = 6$	→	$x_1 = 6$
(2)	$2x_1 + 3x_2 + s_2 = 19$	→	$s_2 = 7$
(3)	$x_1 + x_2 + s_3 = 8$	→	$s_3 = 2$

Setting Up the Simplex Method

- Canonical Form
  - A set of equations is in canonical form (or proper form of Gaussian elimination) if for each equation, its right hand side is non-negative, and there is a single basic variable in the equation.
  - A variable is basic if it appears in only one of the constraint equations, with coefficient 1. The right-hand side of that equation then immediately gives the value of the basic variable.

Setting Up the Simplex Method

- Canonical Form
  - What if the origin is infeasible wrt functional constraints?
    - Introduction of artificial variables
      - Artificial variables are added to all “at-least” and “equal-to” constraints. These artificial variables are then selected as initial basic variables when setting up the simplex method.
      - For “at-most” constraints, the slack variables are a suitable basic variable.
      - For “at least” constraints, the surplus variables cannot be used as basic variables, because the right-hand side would be negative. The artificial variable corresponds to a slack variable, but on “the wrong side” of the constraint.
      - For “equal to” constraints, no immediate basic variable is available. The artificial variable also corresponds to a slack variable, i.e. the deviation from the equality.
  - Artificial variables and infeasibility
    - This means that, as long as any of the artificial variables is non-zero (and basic), the current solution is not acceptable or infeasible.
    - Therefore, in solving the problem, all artificial variables must become non-basic (zero) first.
      - Big-M method: coefficient of +M in the objective function for a minimisation problem.
        coefficient of -M in the objective function for a maximisation problem.
      - Two-phase method: first minimize the sum of all artificial variables, afterwards optimize the original objective function (cfr. Infra)

Setting Up the Simplex Method

- Canonical Form
  - Example Standard Form

Maximize $100 x_1 + 200 x_2$

subject to

$$\begin{aligned} 2x_1 + 3x_2 + s_1 &= 2\,000 \\ x_1 - s_2 &= 60 \\ x_2 + s_3 &= 720 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

Setting Up the Simplex Method

- Canonical Form
  - Example Standard Form

Minimize $100\,x_A + 80\,x_B$

subject to

$$2\,x_A - x_B - s_1 = 0$$
$$x_A + x_B - s_2 = 1\,000$$
$$x_A,\ x_B,\ s_1,\ s_2 \geq 0$$

Algebra of the Simplex Method

- Rewriting the Linear Functions (Jordan-Gauss elimination)
  - Using linear algebra, the *m* constraints can be rewritten such that only one of the *m* basic variables appears in each constraint.
  - To do so, so-called *elementary row operations* can be used:
    - multiplying an equation by a non-zero number
    - adding (subtracting) any other equation, multiplied by a non-zero number
  - Each of the *m* constraints then gives:
    - the value of one of the basic variables
    - an expression of the basic variable in terms of non-basic variables
  - Similarly, using elementary row operations, the objective function can be written as a constraint in terms of non-basic variables. E.g. $Z - 5\,x_1 - 7\,x_2 = 0$
    - No slack variable is needed as the objective function is written as an equation.
    - One equation and one unknown variable is added.
    - Solving the set of functional equations leads to solving the value Z.

Algebra of the Simplex Method

- Rewriting the Linear Functions
  - Example: Initialisation step: Select the origin as initial CP solution
    - Select basic variables: s_1, s_2 and s_3.
    - Non-basic variables: $x_1=0$ and $x_2=0$.
    - Rewrite the constraints and objective function:

$$Z - 5x_1 - 7x_2 = 0$$

s.t.

$s_1 = 6 - x_1$	or	$x_1 + s_1 = 6$
$s_2 = 19 - 2x_1 - 3x_2$	or	$2x_1 + 3x_2 + s_2 = 19$
$s_3 = 8 - x_1 - x_2$	or	$x_1 + x_2 + s_3 = 8$

    - Initial BF solution is (0, 0, 6, 19, 8)

Algebra of the Simplex Method

- Iteration: Optimality Test
  - The revised expression of the objective function (c_i) immediately tells us whether increasing a non-basic variable can improve the objective function value.
    - If not, the current solution is the optimum.
    - If so, increasing that non-basic variable takes us out of the corner point, along the boundary of the feasible region.
  - Example: Iteration 1
    - The Z-row gives the rate of improvement in Z if a variable is increased from zero.
    - x_1: 5 > 0 and x_2: 7 > 0 which indicates that the initial solution is not optimal.

Algebra of the Simplex Method

- Iteration: Movement
  - Direction (= Determine the entering variable): Increasing a non-basic variable affects the values of the basic variables. This can easily be determined from the rewritten constraints.
  - Example: Iteration 1

    $Z = 5\,x_1 + 7\,x_2$

    Increase x_1? Rate of improvement in Z = 5
    Increase x_2? Rate of improvement in Z = 7

    7 > 5, so choose x_2 to increase.

Algebra of the Simplex Method

- Iteration: Movement
  - Stop (= Determine the leaving variable):
    - Go as far as possible without leaving the feasible region.
    - Increasing the entering variable changes the values of some of the basic variables (cfr functional constraints).
  - Example: Iteration 1

		$x_1 = 0$
$x_1 + s_1 = 6$	or	$s_1 = 6$
$2x_1 + 3x_2 + s_2 = 19$	or	$s_2 = 19 - 3\,x_2$
$x_1 + x_2 + s_3 = 8$	or	$s_3 = 8 - x_2$

Algebra of the Simplex Method

- Iteration: Movement
  - Stop (= Determine the leaving variable):
    - Check how far the entering variable can be increased without violating the nonnegativity constraints for the basic variables (cfr. nonnegativity constraints)
    - At a certain value of the non-basic variable, one of the basic variables will become zero (and become a non-basic variable).
  - Example: Iteration 1

$s_1 = 6$	no limit on x_2		
$s_2 = 19 - 3\,x_2$	$x_2 \leq 6\ 1/3$	→	s_2 is leaving variable
$s_3 = 8 - x_2$	$x_2 \leq 8$		

x_2 can be increased just to 6 1/3 at which point s_2 drops to 0. Increasing x_2 beyond would cause s_2 to become negative, which would violate feasibility.

Algebra of the Simplex Method

- Iteration: Solving for the new basic feasible solution
  - Then, a new corner point has been reached. The non-basic variable has become a basic variable (*entering variable) and the basic* variable whose value is zero has become a non-basic variable (*leaving variable).*
  - After selecting the entering variable and determining the leaving variable, the constraints and objective function have to be rewritten in terms of only non-basic variables.
  - Example (cfr. Next slide): Iteration 1
- Then, the same procedure can be repeated.

Algebra of the Simplex Method

- Example: Iteration 1
  - New BF solution

	Initial BF solution	New BF solution
Nonbasic variables:	$x_1 = 0, x_2 = 0$	$x_1 = 0, s_2 = 0$
Basic variables:	$s_1 = 6, s_2 = 19, s_3 = 8$	$s_1 = ?, x_2 = 6\ 1/3, s_3 = ?$

  - Rewrite the constraints and objective function to produce the pattern of coefficients of s_2 (0, 0, 1, 0) as the new coefficients of x_2:

(1)	$Z - 5x_1 - 7x_2$	$= 0$	0
(2)	$x_1 + s_1$	$= 6$	0
(3)	$2x_1 + 3x_2 + s_2$	$= 19$	1
(4)	$x_1 + x_2 + s_3$	$= 8$	0

Algebra of the Simplex Method

- Example: Iteration 1
  - Turn the coefficient of x_2 in eq. (3) into 1 by dividing this equation by 3:

(1)	$Z - 5x_1 - 7x_2$	$= 0$	0
(2)	$x_1 + s_1$	$= 6$	0
(3)	$2/3\,x_1 + x_2 + 1/3\,s_2$	$= 6\ 1/3$	1
(4)	$x_1 + x_2 + s_3$	$= 8$	0

  - Turn the coefficient of x_2 in eqs. (1) and (4) into 0 by the following operations:
    eq. (1) = eq. (1) + 7 x eq. (3)   eq. (4) = eq. (4) – 1 x eq. (3)

(1)	$Z - 1/3\,x_1 + 7/3\,s_2$	$= 44\ 1/3$	0
(2)	$x_1 + s_1$	$= 6$	0
(3)	$2/3\,x_1 + x_2 + 1/3\,s_2$	$= 6\ 1/3$	1
(4)	$1/3\,x_1 - 1/3\,s_2 + s_3$	$= 5/3$	0

> Since $x_1 = 0$ and $s_2 = 0$, new BF solution is (0, 6 1/3, 6, 0, 1 2/3)

Algebra of the Simplex Method

- Optimality Test
  - Example: Iteration 2
    - The Z-row gives the rate of improvement in Z if a variable is increased from zero.
    - x_1: 1/3 > 0 and s_2: -7/3 < 0 which indicates that the initial solution is not optimal.
    - Z can be increased by increasing x_1 and not s_2.

Algebra of the Simplex Method

- Movement
  - Example: Iteration 2
    - Functional constraints

		$s_2 = 0$
$x_1 + s_1 = 6$	or	$s_1 = 6 - x_1$
$2/3\, x_1 + x_2 + 1/3\, s_2 = 6\ 1/3$	or	$x_2 = 6\ 1/3 - 2/3\, x_1$
$1/3\, x_1 - 1/3\, s_2 + s_3 = 5/3$	or	$s_3 = 5/3 - 1/3\, x_1$

- Nonnegativity constraints

$s_1 = 6 - x_1$	$x_1 \leq 6$		
$x_2 = 6\ 1/3 - 2/3\, x_1$	$x_1 \leq 19/2$		
$s_3 = 5/3 - 1/3\, x_1$	$x_1 \leq 5$	→	s_3 is leaving variable

x_1 can be increased just to 5 at which point s_3 drops to 0. Increasing x_1 beyond would cause s_3 to become negative, which would violate feasibility.

Algebra of the Simplex Method

- Example: Iteration 2
  - New BF solution

	Initial BF solution	New BF solution
Nonbasic variables:	$x_1 = 0, x_2 = 6\ 1/3$	$s_3 = 0, s_2 = 0$
Basic variables:	$s_1 = 6, s_2 = 0, s_3 = 5/3$	$s_1 = ?, x_2 = ?, x_1 = 5$

  - Rewrite the constraints and objective function to produce the pattern of coefficients of s_3 (0, 0, 0, 1) as the new coefficients of x_1:

(1) $Z - 1/3\, x_1 + 7/3\, s_2 = 44\ 1/3$ 0
(2) $x_1 + s_1 = 6$ 0
(3) $2/3\, x_1 + x_2 + 1/3\, s_2 = 6\ 1/3$ 0
(4) $1/3\, x_1 - 1/3\, s_2 + s_3 = 5/3$ 1

Algebra of the Simplex Method

- Example: Iteration 2
  - Turn the coefficient of x_1 in eq. (4) into 1 by multiplying this equation by 3:

(1) $Z - 1/3\, x_1 + 7/3\, s_2 = 44\ 1/3$ 0
(2) $x_1 + s_1 = 6$ 0
(3) $2/3\, x_1 + x_2 + 1/3\, s_2 = 6\ 1/3$ 0
(4) $x_1 - s_2 + 3\, s_3 = 5$ 1

  - Turn the coefficient of x_2 in eqs. (1), (2) and (3) into 0 by the following operations:

eq. (1) = eq. (1) + 1/3 x eq. (4)   eq. (2) = eq. (2) - 1 x eq. (4)
eq. (3) = eq. (3) – 2/3 x eq. (4)

(1) $Z + 2\, s_2 + 2\, s_3 = 46$ 0
(2) $s_1 + s_2 - 3\, s_3 = 1$ 0
(3) $x_2 + s_2 - 2\, s_3 = 3$ 0
(4) $x_1 - s_2 + 3\, s_3 = 5$ 1

> Since $x_1 = 5$ and $s_2 = 0$, new BF solution is (5, 3, 1, 0, 0)

The Simplex Method

- Setting up the simplex method (cfr. supra)
  - Step 1: If the problem is a minimization problem, multiply the objective function by -1.
  - Step 2: If the problem formulation contains any constraints with negative right-hand sides, multiply each constraint by -1.
  - Step 3: Put the problem into standard form
    - Add a slack variable to each $\leq$ constraint.
    - Subtract a surplus variable and add an artificial variable to each $\geq$ constraint.
    - Set each slack and surplus variable's coefficient in the objective function to zero.
  - Step 4: Select the origin as initial basic solution
    - Select the decision variables to be the initial nonbasic variables (set equal to zero).

The Simplex Method

- Setting up the simplex method (cfr. Supra)
  - Step 5: Put the problem into canonical form
    - Add an artificial variable to each equality constraint and each $\geq$ constraint.
    - Set each artificial variable's coefficient in the objective function equal to -M, where M is a very large number.
    - Each slack and artificial variable becomes one of the basic variables in the initial basic solution.
      - All basic variables have a coefficient of 1
      - There is 1 basic variable in each constraint
      - Each basic variable appears in 1 constraint
  - Step 6: Rewrite the objective function in terms of non-basic variables only such that the entering basic variable can be easily determined. Hence, eliminate all basic variables from this row using elementary row operations.

The Simplex Method

- Setting up the simplex method
  - The Simplex Tableau Form

	Variables x_i	RHS	
Objective	c_i		Trade ratios
Basis	Exchange coefficients A_{ij}	b_i	

- The coefficients of the variables
- The constants on the right-hand side
- The basic variable in each equation

The Simplex Method

- The Simplex Tableau Form
  - Basis: The list of basic variables in the current solution. All variables not listed are the non-basic variables.
  - C_i: The net effect on the objective of bringing one unit of each variable into the basis
  - Amounts: list of amount of each basic variable and the total contribution of the current solution.
  - Exchange coefficients: The amount of each basic variable in the current solution that must be given up to get one unit on each variable in the linear program
  - Trade ratios: The maximum amounts of the entering variables that can be exchanged for the entire quantity of the basic variables

The Simplex Method

- Perform an iteration of the simplex method
  - Step 1: Determine entering variable
  - Step 2: Determine leaving variable
  - Step 3:Generate next tableau

The Simplex Method

- Step 1: Determine entering variable:
  - Identify the variable with the most negative value in the objective row. (The corresponding column j^* is the pivot column.)
  - If there are no negative values in the objective row, STOP.
    - If there is an artificial variable in the basis with a strict positive value on the RHS, the problem is infeasible.
    - Otherwise, an optimal solution has been found. The current values of the basic variables are optimal. The values of the non-basic variables are all zero.
    - If any non-basic variable's objective row value is 0, alternate optimal solutions might exist.

The Simplex Method

- Step 2: Determine leaving variable
  - For each positive number (> 0) in the pivot column, compute the trade ratio: the right-hand side value divided by the positive exchange coefficient in the pivot column.
  - If there are no positive values in the pivot column, STOP; the problem is unbounded.
  - Otherwise, select the variable with the smallest ratio. The basic for that row is the leaving basic variable. The corresponding row i^* is the pivot row.

The Simplex Method

- Step 3: Generate New Tableau
  - The entering variable replaces the leaving variable in the basic variable column of the next simplex tableau. Solve for the new BF solution by using elementary row operations.
  - Divide the pivot row i^* by the pivot element $A_{i^*j^*}$ *to* get the new row i^* (the entry at the intersection of the pivot row and the pivot column).
  - Replace each non-pivot row i with:

    [new row i] = [current row i] - [(A_{ij^*}) x (row i*)]

    (with a_{ij^*} is the value in entering column j^* of row i)
  - Replace the objective row with:

    [new obj row] = [current obj row] - [(c_{j^*}) x (row i^*)]
  - Return to step 1.

Example A

- Mathematical Problem Formulation

Maximize $1.00\,x_G + 1.35\,x_W$

subject to

$$\begin{aligned} 2\,x_G + 4\,x_W &\leq 500 \\ x_G &\leq 200 \\ x_W &\leq 120 \\ x_G,\ x_W &\geq 0 \end{aligned}$$

- Standard Form

Maximize $1.00\,x_G + 1.35\,x_W$

subject to

$$\begin{aligned} 2\,x_G + 4\,x_W + s_1 &= 500 \\ x_G + s_2 &= 200 \\ x_W + s_3 &= 120 \\ x_G,\ x_W,\ s_1,\ s_2,\ s_3 &\geq 0 \end{aligned}$$

Example A

- The Initial Simplex Tableau

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	-1.35	0	0	0	0
s_1	2	4	1	0	0	500
s_2	1	0	0	1	0	200
s_3	0	1	0	0	1	120

Example A

- Iteration 1
  - Step 1: Determine the Entering Variable

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	-1.35	0	0	0	0
s_1	2	4	1	0	0	500
s_2	1	0	0	1	0	200
s_3	0	1	0	0	1	120

x_W is the variable with the most negative value in the objective row. x_W is the entering variable.

Example A

- Iteration 1
  - Step 2: Determine the Leaving Variable
    - Take the ratio between the right hand side and the positive number in the x_W column

Basic var	x_G	x_W	s_1	s_2	s_3	RHS	
Z	-1.00	-1.35	0	0	0	0	
s_1	2	4	1	0	0	500	500/4 = 125
s_2	1	0	0	1	0	200	-
s_3	0	1	0	0	1	120	120/1 = 120 MIN

s_3 is the variable with the minimal ratio. s_3 is the leaving variable and 1 is the pivot element

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	-1.35	0	0	0	0
s_1	2	4	1	0	0	500
s_2	1	0	0	1	0	200
s_3	0	1	0	0	1	120

Example A

- Iteration 1
  - Step 3: Generate New Tableau
    - Divide the third row (row i^*) by 1 (the pivot element) to get the new row i^*

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	-1.35	0	0	0	0
s_1	2	4	1	0	0	500
s_2	1	0	0	1	0	200
s_3	0	1	0	0	1	120

    - Replace each non-pivot row i with [new row i] = [current row i] - [(A_{ij^*}) x (row i*)]
      [new row 1] = [current row 1] – 4 [row 3]
      [new row 2] = [current row 2] – 0 [row 3]

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	-1.35	0	0	0	0
s_1	2	0	1	0	-4	20
s_2	1	0	0	1	0	200
s_3	0	1	0	0	1	120

Example A

- Iteration 1
  - Step 3: Generate New Tableau
    - Replace the objective row with:
      [new obj row] = [current obj row] – [(-1.35) x (row 3)]

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	0	0	0	1.35	162
s_1	2	0	1	0	-4	20
s_2	1	0	0	1	0	200
x_W	0	1	0	0	1	120

Example A

- Iteration 2
  - Step 1: Determine the Entering Variable

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	0	0	0	1.35	162
s_1	2	0	1	0	-4	20
s_2	1	0	0	1	0	200
x_W	0	1	0	0	1	120

x_G is the variable with the most negative value in the objective row. x_G is the entering variable.

Example A

- Iteration 2
  - Step 2: Determine the Leaving Variable
    - Take the ratio between the right hand side and the positive number in the x_G column

Basic var	x_G	x_W	s_1	s_2	s_3	RHS	
Z	-1.00	0	0	0	1.35	162	
s_1	2	0	1	0	-4	20	20/2 = 10 MIN
s_2	1	0	0	1	0	200	200/1 = 200
x_W	0	1	0	0	1	120	-

s_1 is the variable with the minimal ratio. s_1 is the leaving variable and 2 is the pivot element

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	0	0	0	1.35	162
s_1	2	0	1	0	-4	20
s_2	1	0	0	1	0	200
x_W	0	1	0	0	1	120

Example A

- Iteration 2
  - Step 3: Generate New Tableau
    - Divide the first row (row i^*) by 2 (the pivot element) to get the new row i^*

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	0	0	0	1.35	162
s_1	1	0	1/2	0	-2	10
s_2	1	0	0	1	0	200
x_W	0	1	0	0	1	120

    - Replace each non-pivot row i with [new row i] = [current row i] - [(A_{ij^*}) x (row i*)]

      [new row 2] = [current row 2] – 1 [row 1]

      [new row 3] = [current row 3] – 0 [row 1]

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	-1.00	0	0	0	1.35	162
s_1	1	0	1/2	0	-2	10
s_2	0	0	-1/2	1	2	190
x_W	0	1	0	0	1	120

Example A

- Iteration 2
  - Step 3: Generate New Tableau
    - Replace the objective row with:

      [new obj row] = [current obj row] – [(-1.00) x (row 1)]

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	0	0	1/2	0	-0.65	172
x_G	1	0	1/2	0	-2	10
s_2	0	0	-1/2	1	2	190
x_W	0	1	0	0	1	120

Example A

- Iteration 3
  - Step 1: Determine the Entering Variable

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	0	0	1/2	0	-0.65	172
x_G	1	0	1/2	0	-2	10
s_2	0	0	-1/2	1	2	190
x_W	0	1	0	0	1	120

s_3 is the variable with the most negative value in the objective row. s_3 is the entering variable.

Example A

- Iteration 3
  - Step 2: Determine the Leaving Variable
    - Take the ratio between the right hand side and the positive number in the s_3 column

Basic var	x_G	x_W	s_1	s_2	s_3	RHS	
Z	0	0	1/2	0	-0.65	172	
x_G	1	0	1/2	0	-2	10	-
s_2	0	0	-1/2	1	2	190	190/2 = 95 MIN
x_W	0	1	0	0	1	120	120/1 = 120

s_2 is the variable with the minimal ratio. s_2 is the leaving variable and 2 is the pivot element

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	0	0	1/2	0	-0.65	172
x_G	1	0	1/2	0	-2	10
s_2	0	0	-1/2	1	2	190
x_W	0	1	0	0	1	120

Example A

- Iteration 3
  - Step 3: Generate New Tableau
    - Divide the second row (row i^*) by 2 (the pivot element) to get the new row i^*

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	0	0	1/2	0	-0.65	172
x_G	1	0	1/2	0	-2	10
s_2	0	0	-1/4	1/2	1	95
x_W	0	1	0	0	1	120

    - Replace each non-pivot row i with [new row i] = [current row i] - [(A_{ij^*}) x (row i*)]

      [new row 1] = [current row 1] + 2 [row 2]

      [new row 3] = [current row 3] – 1 [row 2]

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	0	0	1/2	0	-0.65	172
x_G	1	0	0	1	0	200
s_2	0	0	-1/4	1/2	1	95
x_W	0	1	1/4	-1/2	0	25

Example A

- Iteration 3
  - Step 3: Generate New Tableau
    - Replace the objective row with:

      [new obj row] = [current obj row] – [0.65 x (row 2)]

Basic var	x_G	x_W	s_1	s_2	s_3	RHS
Z	0	0	27/80	13/40	0	233.75
x_G	1	0	0	1	0	200
s_2	0	0	-1/4	1/2	1	95
x_W	0	1	1/4	-1/2	0	25

  - Since there are no negative numbers in the objective row, this tableau is optimal.
    - The optimal solution is (x_G, x_W, s_1, s_2, s_3) = (200, 25, 0, 95, 0)
    - The optimal value of the objective function is 233.75.

Example B

- Mathematical Problem Formulation

  Maximize $100x_1 + 200x_2$

  subject to

  $$\begin{aligned} 2x_1 + 3x_2 &\le 2\,000 \\ x_1 &\ge 60 \\ x_2 &\le 720 \\ x_1, x_2 &\ge 0 \end{aligned}$$

  - Canonical Form

  Maximize $100x_1 + 200x_2 - Ma_1$

  subject to

  $$\begin{aligned} 2x_1 + 3x_2 + s_1 &= 2\,000 \\ x_1 - s_2 + a_1 &= 60 \\ x_2 + s_3 &= 720 \\ x_1, x_2, s_1, s_2, s_3, a_1 &\ge 0 \end{aligned}$$

Example B

- The Initial Simplex Tableau

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	-100	-200	0	0	0	M	0
s_1	2	3	1	0	0	0	2000
a_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

  - The basic variable a_1 has a nonzero coefficient in the Z-row. All basic variables should be eliminated from the Z-row before the simplex method can be applied using the following transformation:

$$\begin{array}{lll} & Z - 100x_1 - 200x_2 + Ma_1 & = 0 \\ -M(& x_1 + \quad - \quad s_2 + a_1 & = 60) \\ \hline & Z - (-100 - M)x_1 - 200x_2 + M s_2 & = -60M \end{array}$$

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	-M-100	-200	0	M	0	0	-60M
s_1	2	3	1	0	0	0	2000
a_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

Example B

- Iteration 1
  - Step 1: Determine the Entering Variable

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	-M-100	-200	0	M	0	0	-60M
s_1	2	3	1	0	0	0	2000
a_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

x_1 is the variable with the most negative value in the objective row. x_1 is the entering variable.

Example B

- Iteration 1
  - Step 2: Determine the Leaving Variable
    - Take the ratio between the right hand side and the positive number in the x_1 column

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS	
Z	-M-100	-200	0	M	0	0	-60M	
s_1	2	3	1	0	0	0	2000	2000/2 = 1000
a_1	1	0	0	-1	0	1	60	60/1 = 60 MIN
s_3	0	1	0	0	1	0	720	-

a_1 is the variable with the minimal ratio. a_1 is the leaving variable and 1 is the pivot element

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	-M-100	-200	0	M	0	0	-60M
s_1	2	3	1	0	0	0	2000
a_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

Example B

- Iteration 1
  - Step 3: Generate New Tableau
    - Divide the second row (row i^*) by 1 (the pivot element) to get the new row i^*

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	-M-100	-200	0	M	0	0	-60M
s_1	2	3	1	0	0	0	2000
a_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

    - Replace each non-pivot row i with [new row i] = [current row i] - [(A_{ij^*}) x (row i*)]
      [new row 1] = [current row 1] – 2 [row 2]
      [new row 3] = [current row 3] – 0 [row 2]

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	-M-100	-200	0	M	0	0	-60M
s_1	0	3	1	2	0	-2	1880
a_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

Example B

- Iteration 1
  - Step 3: Generate New Tableau
    - Replace the objective row with:
      [new obj row] = [current obj row] – [(-M-100) x (row 2)]

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	0	-200	0	-100	0	M+100	6000
s_1	0	3	1	2	0	-2	1880
x_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

Example B

- Iteration 2
  - Step 1: Determine the Entering Variable

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	0	-200	0	-100	0	M+100	6000
s_1	0	3	1	2	0	-2	1880
x_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

x_2 is the variable with the most negative value in the objective row. x_2 is the entering variable.

Example B

- Iteration 2
  - Step 2: Determine the Leaving Variable
    - Take the ratio between the right hand side and the positive number in the x_1 column

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS	
Z	0	-200	0	-100	0	M+100	6000	
s_1	0	3	1	2	0	-2	1880	1880/3 = 626.66 MIN
x_1	1	0	0	-1	0	1	60	-
s_3	0	1	0	0	1	0	720	720/1 = 720

s_1 is the variable with the minimal ratio. s_1 is the leaving variable and 3 is the pivot element

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	0	-200	0	-100	0	M+100	6000
s_1	0	3	1	2	0	-2	1880
x_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

Example B

- Iteration 2
  - Step 3: Generate New Tableau
    - Divide the first row (row i^*) by 3 (the pivot element) to get the new row i^*

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	0	-200	0	-100	0	M+100	6000
s_1	0	1	1/3	2/3	0	-2/3	1880/3
x_1	1	0	0	-1	0	1	60
s_3	0	1	0	0	1	0	720

    - Replace each non-pivot row i with [new row i] = [current row i] - [(A_{ij^*}) x (row i*)]

      [new row 2] = [current row 2] – 0 [row 1]

      [new row 3] = [current row 3] – 1 [row 1]

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	0	-200	0	-100	0	M+100	6000
s_1	0	1	1/3	2/3	0	-2/3	1880/3
x_1	1	0	0	-1	0	1	60
s_3	0	0	-1/3	-2/3	1	2/3	280/3

Example B

- Iteration 2
  - Step 3: Generate New Tableau
    - Replace the objective row with:

      [new obj row] = [current obj row] – [(-200) x (row 2)]

Basic var	x_1	x_2	s_1	s_2	s_3	a_1	RHS
Z	0	0	200/3	100/3	0	M-100/3	394000/3
x_2	0	1	1/3	2/3	0	-2/3	1880/3
x_1	1	0	0	-1	0	1	60
s_3	0	0	-1/3	-2/3	1	2/3	280/3

  - Since there are no negative numbers in the objective row, this tableau is optimal.
    - The optimal solution is (x_1, x_2, s_1, s_2, s_3, a_1) = (60, 1880/3, 0, 0, 280/3, 0)
    - The optimal value of the objective function is 394 000/3.

Example C

- Simplex Tableau

$$\begin{aligned} Z - 3x_1 - 5x_2 \quad &= 0 \\ x_1 + \quad s_1 \quad &= 4 \\ 2x_2 + \quad s_2 \quad &= 12 \\ 3x_1 + 2x_2 + \quad s_3 \quad &= 18 \\ x_1, \quad x_2, \quad s_1, \; s_2, \; s_3 \quad &\geq 0 \end{aligned}$$

Basic var	Z	x_1	x_2	s_1	s_2	s_3	RHS
Z	1	-3	-5	0	0	0	0
s_1	0	1	0	1	0	0	4
s_2	0	0	2	0	1	0	12
s_3	0	3	2	0	0	1	18

Special Cases

- Tie for the entering variable
  - In step 1, if two or more non-basic variables are tied for having the most negative coefficient in the objective row, the entering basic variable may be chosen arbitrarily among these variables.
- Tie for the leaving variable
  - In step 2, if two or more basic variables tie for having the smallest trade ratio, the leaving basic variable may be chosen arbitrarily among these variables. The other variable (that remains in the basis) will also become zero in the new BF solution.
- Degeneracy
  - A basic variable with a value of zero is called a degenerate variable; a solution with a degenerate variable is called a degenerate solution.
  - This can occur at formulation or if there is a tie for the minimising value in the ratio test to determine the leaving variable.

Special Cases

- Alternative Optimal Solutions
  - If there is a non-basic variable with an objective row value equal to zero in the final tableau, there are multiple optima available.
- Unboundedness
  - If all entries in an entering column are non-positive, there is no leaving variable.
  - In that case, the entering basic variable could be increased indefinitely without giving negative values to any of the current basic variables and the problem is unbounded.
- Infeasibility
  - If there is an artificial variable in the optimal solution (i.e. the artifical variable remains positive in the final tableau), the problem is infeasible .

Example D: Degeneracy

- Mathematical Problem Formulation

Maximize $12x_1 + 12x_2 + 10x_3$

subject to

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &\leq 50 \\ -x_1 + x_2 - x_3 &\geq 0 \\ x_1 + x_3 &\geq 0 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

  - Canonical Form

Maximize $12x_1 + 12x_2 + 10x_3$

subject to

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 + s_1 &= 50 \\ x_1 - x_2 + x_3 + s_2 &= 0 \\ -x_1 - x_3 + s_3 &= 0 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

Example E: Alternative Optimal Solutions

- Mathematical Problem Formulation

Maximize $3x_1 + 2x_2$

subject to

$$
\begin{aligned}
x_1 \quad\quad & \leq 4 \\
2x_2 & \leq 12 \\
3x_1 + 2x_2 & \leq 18 \\
x_1, \quad x_2 & \geq 0
\end{aligned}
$$

- Canonical Form

Maximize $3x_1 + 2x_2$

subject to

$$
\begin{aligned}
x_1 \quad\quad + s_1 \quad\quad\quad & = 4 \\
2x_2 + \quad s_2 \quad & = 12 \\
3x_1 + 2x_2 + \quad\quad s_3 & = 18 \\
x_1, \quad x_2, \quad s_1, \; s_2, \quad s_3 & \geq 0
\end{aligned}
$$

Example F: Unbounded Problem

- Mathematical Problem Formulation

Maximize $2x_1 + 6x_2$

subject to

$$
\begin{aligned}
4x_1 + 3x_2 & \geq 12 \\
2x_1 + x_2 & \geq 8 \\
x_1, \quad x_2 & \geq 0
\end{aligned}
$$

- Canonical Form

Maximize $2x_1 + 6x_2 - Ma_1 - Ma_2$

subject to

$$
\begin{aligned}
4x_1 + 3x_2 - s_1 \quad + \; a_1 \quad\quad & = 12 \\
2x_1 + x_2 - \quad s_2 \quad + \; a_2 & = 8 \\
x_1, \quad x_2, \quad s_1, \; s_2 & \geq 0
\end{aligned}
$$

Example G: Infeasible Problem

- Mathematical Problem Formulation

  Maximize $2x_1 + 6x_2$

  subject to

  $4x_1 + 3x_2 \leq 12$

  $2x_1 + x_2 \geq 8$

  $x_1, x_2 \geq 0$

  - Canonical Form

    Maximize $2x_1 + 6x_2 - Ma_2$

    subject to

    $4x_1 + 3x_2 + s_1 = 12$

    $2x_1 + x_2 - s_2 + Ma_2 = 8$

    $x_1, x_2, s_1, s_2, s_3 \geq 0$

Two-Phase Method

- When using the big M-method, we can split the problem and solve the problem in two phases.
  - Phase 1: Divide the big M method objective function terms by M and drop the other negligible terms.

    Minimize Z = ∑ artificial variables

    subject to Revised constraints (with artificial variables)

  - Phase 2: Find the optimal solution for the real problem. Use the optimal solution of phase 1 as initial basic feasible solution for applying the simplex method to the real problem (the big M method coefficients can be dropped dependent of outcome of phase 1).

    Minimize Z = ∑ original variables

    subject to Original constraints (without artificial variables)

  - This approach is justified as the M-terms dominate the negligible terms.

Example H: Two-Phase Method

- Mathematical Problem Formulation
  - LP formulation

Minimize $2x_1 + 3x_2 + x_3$

subject to
$$x_1 + 4x_2 + 2x_3 \geq 8$$
$$3x_1 + 2x_2 \geq 6$$
$$x_1, \; x_2, \; x_3 \geq 0$$

  - Canonical Form (Big M-method)

Minimize $2x_1 + 3x_2 + x_3 + Ma_1 + Ma_2$

subject to
$$x_1 + 4x_2 + 2x_3 - s_1 + a_1 = 8$$
$$3x_1 + 2x_2 - s_2 + a_2 = 6$$
$$x_1, \; x_2, \; x_3, \; s_1, \; s_2, \; a_1, \; a_2 \geq 0$$

Example H: Two-Phase Method

- Mathematical Problem Formulation Two-phase Method
  - Phase 1 Problem

Minimize $a_1 + a_2 \Leftrightarrow$ Max $-a_1 - a_2$

subject to
$$x_1 + 4x_2 + 2x_3 - s_1 + a_1 = 8$$
$$3x_1 + 2x_2 - s_2 + a_2 = 6$$
$$x_1, \; x_2, \; x_3, \; s_1, \; s_2, \; a_1, \; a_2 \geq 0$$

  - Phase 2 Problem

Minimize $2x_1 + 3x_2 + x_3 \Leftrightarrow$ Max $-2x_1 - 3x_2 - x_3$

subject to
$$x_1 + 4x_2 + 2x_3 - s_1 = 8$$
$$3x_1 + 2x_2 - s_2 = 6$$
$$x_1, \; x_2, \; x_3, \; s_1, \; s_2 \geq 0$$

Example H: Two-Phase Method

- Phase 1: The Initial Simplex Tableau

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	0	0	0	0	0	1	1	0
a_1	1	4	2	-1	0	1	0	8
a_2	3	2	0	0	-1	0	1	6

  - The basic variables a_1 and a_2 have a nonzero coefficient in the Z-row. All basic variables should be eliminated from the Z-row before the simplex method can be applied using the following transformation:

$$\begin{array}{llllll} & -Z & & & +a_1 + a_2 & = 0 \\ -1(& x_1 + 4x_2 + 2x_3 & -s_1 & & +a_1 & = 8) \\ -1(& 3x_1 + 2x_2 & & -s_2 & + a_2 & = 6) \\ \hline & Z - 4x_1 - 6x_2 - 2x_3 + s_1 & & + s_2 & & = -14 \end{array}$$

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-4	-6	-2	1	1	0	0	-14
a_1	1	4	2	-1	0	1	0	8
a_2	3	2	0	0	-1	0	1	6

Example H: Two-Phase Method

- Phase 1: Iteration 1
  - Step 1: Determine the Entering Variable

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-4	-6	-2	1	1	0	0	-14
a_1	1	4	2	-1	0	1	0	8
a_2	3	2	0	0	-1	0	1	6

x_2 is the variable with the most negative value in the objective row. x_2 is the entering variable.

Example H: Two-Phase Method

- Phase 1: Iteration 1
  - Step 2: Determine the Leaving Variable
    - Take the ratio between the right hand side and the positive number in the x_2 column

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS	
- Z	-4	-6	-2	1	1	0	0	-14	
a_1	1	4	2	-1	0	1	0	8	8/4 = 2 MIN
a_2	3	2	0	0	-1	0	1	6	6/2 = 3

a_1 is the variable with the minimal ratio. a_1 is the leaving variable and 4 is the pivot element

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-4	-6	-2	1	1	0	0	-14
a_1	1	4	2	-1	0	1	0	8
a_2	3	2	0	0	-1	0	1	6

Example H: Two-Phase Method

- Phase 1: Iteration 1
  - Step 3: Generate New Tableau
    - Divide the first row (row i^*) by 4 (the pivot element) to get the new row i^*

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-4	-6	-2	1	1	0	0	-14
a_1	1/4	1	1/2	-1/4	0	1/4	0	2
a_2	3	2	0	0	-1	0	1	6

    - Replace each non-pivot row i with [new row i] = [current row i] - [(A_{ij^*}) x (row i*)]

[new row 2] = [current row 2] – 2 [row 1]

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-4	-6	-2	1	1	0	0	-14
a_1	1/4	1	1/2	-1/4	0	1/4	0	2
a_2	5/2	0	-1	1/2	-1	-1/2	1	2

Example H: Two-Phase Method

- Phase 1: Iteration 1
  - Step 3: Generate New Tableau
    - Replace the objective row with:

      [new obj row] = [current obj row] – [(-6) x (row 1)]

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-5/2	0	1	-1/2	1	3/2	0	-2
x_2	1/4	1	1/2	-1/4	0	1/4	0	2
a_2	5/2	0	-1	1/2	-1	-1/2	1	2

Example H: Two-Phase Method

- Phase 1: Iteration 2
  - Step 1: Determine the Entering Variable

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-5/2	0	1	-1/2	1	3/2	0	-2
x_2	1/4	1	1/2	-1/4	0	1/4	0	2
a_2	5/2	0	-1	1/2	-1	-1/2	1	2

x_1 is the variable with the most negative value in the objective row. x_1 is the entering variable.

Example H: Two-Phase Method

- Phase 1: Iteration 2
  - Step 2: Determine the Leaving Variable
    - Take the ratio between the right hand side and the positive number in the x_1 column

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS	
- Z	-5/2	0	1	-1/2	1	3/2	0	-2	
x_2	1/4	1	1/2	-1/4	0	1/4	0	2	2/(1/4) = 8
a_2	5/2	0	-1	1/2	-1	-1/2	1	2	2/(5/2) = 4/5 MIN

a_2 is the variable with the minimal ratio. a_2 is the leaving variable and 5/2 is the pivot element

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-5/2	0	1	-1/2	1	3/2	0	-2
x_2	1/4	1	1/2	-1/4	0	1/4	0	2
a_2	5/2	0	-1	1/2	-1	-1/2	1	2

Example H: Two-Phase Method

- Phase 1: Iteration 2
  - Step 3: Generate New Tableau
    - Divide the second row (row i^*) by 5/2 (the pivot element) to get the new row i^*

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-5/2	0	1	-1/2	1	3/2	0	-2
x_2	1/4	1	1/2	-1/4	0	1/4	0	2
a_2	1	0	-2/5	1/5	-2/5	-1/5	2/5	4/5

    - Replace each non-pivot row i with [new row i] = [current row i] - [(A_{ij^*}) x (row i*)]

[new row 1] = [current row 1] – (1/4) [row 2]

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	-5/2	0	1	-1/2	1	3/2	0	-2
x_2	0	1	3/5	-3/10	1/10	3/10	-1/10	9/5
a_2	1	0	-2/5	1/5	-2/5	-1/5	2/5	4/5

Example H: Two-Phase Method

- Phase 1: Iteration 2
  - Step 3: Generate New Tableau
    - Replace the objective row with:

      [new obj row] = [current obj row] – [(-5/2) x (row 1)]

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	0	0	0	0	0	1	1	0
x_2	0	1	3/5	-3/10	1/10	3/10	-1/10	9/5
x_1	1	0	-2/5	1/5	-2/5	-1/5	2/5	4/5

  - Since there are no negative numbers in the objective row, this tableau is optimal.
  - Note that there are different non-basic variables with an objective row value equal to zero in the final tableau. There are multiple optimal solutions available.

Example H: Two-Phase Method

- Phase 2: The Initial Simplex Tableau

Basic var	x_1	x_2	x_3	s_1	s_2	a_1	a_2	RHS
- Z	0	0	0	0	0	1	1	0
x_2	0	1	3/5	-3/10	1/10	3/10	-1/10	9/5
x_1	1	0	-2/5	1/5	-2/5	-1/5	2/5	4/5

  - Use optimal solution of phase 1 as initial solution for phase 2 by dropping the columns of the artifical variables.

Basic var	x_1	x_2	x_3	s_1	s_2	RHS
- Z	0	0	0	0	0	0
x_2	0	1	3/5	-3/10	1/10	9/5
x_1	1	0	-2/5	1/5	-2/5	4/5

  - Sustitute phase 2 objective function

Basic var	x_1	x_2	x_3	s_1	s_2	RHS
- Z	2	3	1	0	0	0
x_2	0	1	3/5	-3/10	1/10	9/5
x_1	1	0	-2/5	1/5	-2/5	4/5

Example H: Two-Phase Method

- Phase 2: The Initial Simplex Tableau
  - The basic variables x_1 and x_2 have a nonzero coefficient in the Z-row. All basic variables should be eliminated from the Z-row before the simplex method can be applied using the following transformation:

$$
\begin{array}{lllllll}
 & Z + 2x_1 + 3x_2 + x_3 & & & & & = 0 \\
-3(& & x_2 + 3/5x_3 & -3/10 s_1 & +1/10 s_2 & & = 9/5) \\
-2(& x_1 & -2/5x_3 & +1/5 s_1 & -2/5 s_2 & & = 4/5) \\
\hline
 & Z & & +1/2 s_1 & +1/2 s_2 & & = -7
\end{array}
$$

Basic var	x_1	x_2	x_3	s_1	s_2	RHS
- Z	0	0	0	1/2	1/2	-7
x_2	0	1	3/5	-3/10	1/10	9/5
x_1	1	0	-2/5	1/5	-2/5	4/5

  - Since there are no negative numbers in the objective row, this tableau is optimal and Z = 7.